Diploma Programme
Programme du diplôme
Programa del Diploma

Further mathematics
Higher level
Paper 2

Thursday 21 May 2015 (morning)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

2215 – 7102

5 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 20]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + y - 1$ with boundary condition $y = 1$ when $x = 0$.

(a) Using Euler's method with increments of 0.2, find an approximate value for y when $x = 1$. [5]

(b) Explain how Euler's method could be improved to provide a better approximation. [1]

(c) Solve the differential equation to find an exact value for y when $x = 1$. [9]

(d) (i) Find the first three non-zero terms of the Maclaurin series for y.

(ii) Hence find an approximate value for y when $x = 1$. [5]

2. [Maximum mark: 13]

In a large population of sheep, their weights are normally distributed with mean $\mu\,\mathrm{kg}$ and standard deviation $\sigma\,\mathrm{kg}$. A random sample of 100 sheep is taken from the population. The mean weight of the sample is $\bar{X}$ kg.

(a) State the distribution of $\bar{X}$, giving its mean and standard deviation. [2]

(b) The sample values are summarized as $\sum x = 3782$ and $\sum x^2 = 155341$ where x kg is the weight of a sheep.

(i) Find unbiased estimates for μ and σ^2.

(ii) Find a 95% confidence interval for μ. [6]

(c) Test, at the 1% level of significance, the null hypothesis $\mu = 35$ against the alternative hypothesis that $\mu > 35$. [5]

3. [Maximum mark: 18]

In 1985, the deer population in a national park was 330. A year later it had increased to 420. To model these data the year 1985 was designated as year zero. The increase in deer population from year $n-1$ to year n is three times the increase from year $n-2$ to year $n-1$. The deer population in year n is denoted by x_n.

(a) Show that for $n \ge 2$, $x_n = 4x_{n-1} - 3x_{n-2}$. [3]

(b) Solve the recurrence relation. [6]

(c) Show using proof by strong induction that the solution is correct. [9]

4. [Maximum mark: 21]

Consider the ellipse having equation $x^2 + 3y^2 = 2$.

(a) (i) Find the equation of the tangent to the ellipse at the point $\left(1, \frac{1}{\sqrt{3}}\right)$.

(ii) Find the equation of the normal to the ellipse at the point $\left(1, \frac{1}{\sqrt{3}}\right)$. [7]

(b) Given that the tangent crosses the x-axis at P and the normal crosses the y-axis at Q, find the equation of (PQ). [4]

(c) Hence show that (PQ) touches the ellipse. [4]

(d) State the coordinates of the point where (PQ) touches the ellipse. [1]

(e) Find the coordinates of the foci of the ellipse. [4]

(f) Find the equations of the directrices of the ellipse. [1]

5. [Maximum mark: 19]

(a) By considering the points $(1, 0)$ and $(0, 1)$ determine the 2×2 matrix which represents

(i) an anticlockwise rotation of θ about the origin;

(ii) a reflection in the line $y = (\tan\theta)x$. [5]

(b) Determine the matrix A which represents a rotation from the direction $\begin{pmatrix}1\\0\end{pmatrix}$ to the direction $\begin{pmatrix}1\\3\end{pmatrix}$. [2]

(c) A triangle whose vertices have coordinates $(0, 0)$, $(3, 1)$ and $(1, 5)$ undergoes a transformation represented by the matrix $A^{-1}XA$, where X is the matrix representing a reflection in the x-axis. Find the coordinates of the vertices of the transformed triangle. [6]

(d) The matrix $B = A^{-1}XA$ represents a reflection in the line $y = mx$. Find the value of m. [6]

6. [Maximum mark: 15]

Gillian is throwing a ball at a target. The number of throws she makes before hitting the target follows a geometric distribution, $X \sim \text{Geo}(p)$. When she uses a cricket ball the number of throws she makes follows a geometric distribution with $p = \frac{1}{4}$. When she uses a tennis ball the number of throws she makes follows a geometric distribution with $p = \frac{3}{4}$. There is a box containing a large number of balls, 80% of which are cricket balls and the remainder are tennis balls. The random variable A is the number of throws needed to hit the target when a single ball is chosen at random from this box and used for all throws.

(a) Find $\text{E}(A)$. [4]

(b) Show that $\text{P}(A = r) = \frac{1}{5} \times \left(\frac{3}{4}\right)^{r-1} + \frac{3}{20} \times \left(\frac{1}{4}\right)^{r-1}$. [4]

(c) Find $\text{P}(A \le 5 \mid A > 3)$. [7]

7. [Maximum mark: 16]

S is defined as the set of all 2×2 non-singular matrices. A and B are two elements of the set S.

(a) (i) Show that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

(ii) Show that $(AB)^T = B^T A^T$. [8]

(b) A relation R is defined on S such that A is related to B if and only if there exists an element X of S such that $XAX^T = B$. Show that R is an equivalence relation. [8]

8. [Maximum mark: 15]

(a) Using a Taylor series, find a quadratic approximation for $f(x) = \sin x$ centred about $x = \frac{3\pi}{4}$. [4]

(b) When using this approximation to find angles between 130° and 140°, find the maximum value of the Lagrange form of the error term. [7]

(c) Hence find the largest number of decimal places to which $\sin x$ can be estimated for angles between 130° and 140°. [1]

(d) Explain briefly why the same maximum value of error term occurs for $g(x) = \cos x$ centred around $\frac{\pi}{4}$ when finding approximations for angles between 40° and 50°. [3]

9. [Maximum mark: 13]

Let f be a homomorphism of a group G onto a group H.

(a) Show that if e is the identity in G, then $f(e)$ is the identity in H. [2]

(b) Show that if x is an element of G, then $f\left(x^{-1}\right) = \left(f(x)\right)^{-1}$. [2]

(c) Show that if G is Abelian, then H must also be Abelian. [5]

(d) Show that if S is a subgroup of G, then $f(S)$ is a subgroup of H. [4]